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This Journal is dedicated to the following aims:

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.
2. To supply an additional medium for the publication of expository mathematical articles.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

A Phase of Mathematical Research

In his *Historical Introduction to Mathematical Literature,* G. A. Miller in 1916 wrote as follows: "It would be very conservative to state that the first decade and a half of the present century produced at least one-fifth as much (mathematical literature) as all the preceding centuries combined. Hence, it appears likely that the twentieth will produce, as the nineteenth century has done, much more *new* mathematical literature than the total existing mathematical literature at its beginning."

One would scarcely hazard the prediction that a similar ratio of mathematical production may not hold for the 21st century as compared with all previous ones. If the present rate of world-wide output is maintained, it might seem not altogether impossible that for a long period in the immediate future each century shall be marked by as great an amount of mathematical activity as may mark the previous centuries combined.

The arrival of such a state of affairs may appear highly improbable to those disposed to look with pessimistic eye upon present growing currents of anti-mathematics propaganda. Such pessimists (?) are prone to think that the public and widespread corrosion of the elementary school mathematics foundations must ultimately reduce mathematical output at the higher levels.

But whether one agrees with the optimist who, with prophetic eye, sees higher and yet higher peaks of mathematics rising with the passing centuries; or with the pessimist who envisions the possibility of an era when mathematicians shall be as much out of "fashion"* as out of the schools, from the solid ground of fact, one may still ask: Will the present rapidly increasing volume of mathematical output finally mark a time when it shall be impossible for one to know with certainty whether his product is new, or is a mere duplication of another's previously published or unpublished discovery?

The very existence of such a situation when it shall arrive will automatically affect the traditional conception of mathematical research, namely, that it is a body of mathematical ideas which satisfies along with other conditions that of not having been previously discovered. For, if priority of creation (or discovery) in mathematics is always to be assumed an essential of research, sooner or later we must be faced with one or both of two large consequences: (a) a greatly increased difficulty in establishing the priority of a mathematical achievement will sensibly reduce the amount of published mathematical product; (b) actual mathematical study will be appreciably curtailed, since ordinariiy so much of it is motivated by the desire of the mathematician to do something not only worthwhile, but something not previously done by another.

S. T. SANDERS.

*See *Fashions in Mathematics* by D. R. Curtiss, November issue *American Mathematical Monthly.*

Hyperbolic Solution of the Cubic Equation

By W. T. SHORT
Oklahoma Baptist University

It seems not to be generally known that the hyperbolic functions are useful in solving cubic equations. I have been using these functions in my course in Theory of Equations for the past two years and find them very helpful. I am indebted to Professor Charles Hutchinson of the University of Colorado for a knowledge of the fact that their use is not new.*

I am here giving the steps by which I developed the method. On page 49 of Dickson's "*First Course in the Theory of Equations*" is given the trigonometric solution of the reduced cubic equation with $\Delta > 0$. The results though not expressly so stated are equivalent to the following:

To solve the equation $y^3 + py + q = 0$,

set $$n^2 = \frac{-4p}{3} \text{ and } \cos A = \frac{-4q}{n^3}.$$

Then $y_1 = n \cos A$, $y_2 = n \cos (A + 120°)$, and $y_3 = n \cos (A + 240°)$.

The above substitution fails in the case where $\Delta < 0$. In that case

$$R = \frac{p^3}{27} + \frac{q^2}{4} \text{ is positive.}$$

If R is positive when p is negative, then $\dfrac{q^2}{4} > \left| \dfrac{p^3}{27} \right|$

and since $$\left| \cos A \right| = \sqrt{\frac{q^2}{4} \div \frac{p^3}{27}}$$

the absolute value of $\cos A$ would be greater than 1. If R is positive because p is positive, then the value for

$$n^2 = \frac{-4}{3}$$

would be negative and n would be imaginary.

*See "*Smithsonian Mathematical Formula and Tables of Elliptic Functions*", E. P. Adams, Washington, the Smithsonian Institution, 1922, Publication 2672, Section 1.272, pages 9-10.

There is a substitution which will work for each case and which gives a solution very similar to the trigonometric solution.

In the case where p is negative and R is positive we write the hyperbolic identity:

$$\cosh 3A = 4\cosh^3 A - 3\cosh A$$

in the form

$$z^3 - \frac{3z}{4} - \frac{1}{4}\cosh 3A = 0. \quad z(=\cosh A).$$

Then in the given equation $y^3+py+q=0$, take $y=nz$, and

$$z^3 + \frac{pz}{n^2} + \frac{q}{n^3} = 0$$

will be identical with the former equation in z if

$$n^2 = \frac{-4p}{3} \quad \text{and} \quad \cosh 3A = \frac{-4q}{n^3}.$$

Choose the sign of n to be opposite the sign of q. Then $\cosh 3A$ is greater than 1, and we can find its value from a table of hyperbolic functions. Since $z=\cosh A$, $y=n\cosh A$. If in the equation

$$y^3+py+q=0 \text{ we substitute } p = \frac{-3n^2}{4} \text{ and } q = \frac{-n^3\cosh 3A}{4}$$

and solve for y, we have

$$y_1 = n\cosh A,$$

$$2y_2 = -n\cosh A + n\sinh A i\sqrt{3}, \text{ and}$$

$$2y_3 = -n\cosh A - n\sinh A i\sqrt{3}.$$

Now in the second case where p is positive and R is positive we write the hyperbolic identity:

$$\sinh 3A = 4\sinh^3 A + 3\sinh A$$

in the form $\quad z^3 + \dfrac{3z}{4} - \dfrac{1}{4}\sinh 3A = 0. \quad (z=\sinh A).$

Then in the given equation $y^3+py+q=0$, take $y=nz$ and

$$z^3 + \frac{pz}{n^2} + \frac{q}{n^3} = 0$$

will be identical with the former equation in z if

$$n^2=\frac{4p}{3} \quad \text{and} \quad \sinh 3A=\frac{-4q}{n^3}.$$

Since p is positive, n is real. Choose sign of n to be opposite sign of q. Then $\sinh 3A$ may have any positive value. Therefore we can find the value of $3A$ from a table of hyperbolic functions. Since $z=\sinh A$, $y=n \sinh A$.

If in the equation $y^3+py+q=0$ we substitute

$$p=\frac{3n^2}{4} \quad \text{and} \quad q=\frac{-n^3 \sinh 3A}{4}$$

and solve for y we have

$$y_1=n \sinh A,$$
$$2y_2=-n \sinh A+n \cosh Ai\sqrt{3},$$
$$2y_3=-n \sinh A-n \cosh Ai\sqrt{3}.$$

SUMMARY

To solve the cubic equation $y^3+py+q=0$:

Define $n^2=\left|\frac{4p}{3}\right|$. Choose sign of n to be opposite sign of q.

I. p is negative and $\left|\frac{4q}{n^3}\right|<1$.

Set $n^2=\frac{-4p}{3}$ and $\cos 3A=\frac{-4q}{n^3}$.

Then $y_1=n \cos A$,

$y_2=n \cos(A+120°)=-n \sin(A+30°)$ and

$y_3=n \cos(A+240°)=-n \cos(A+60°)$.

II. p is negative and $\left|\frac{4q}{n^3}\right|>1$.

Set $n^2=\frac{-4p}{3}$ and $\cosh 3A=\frac{-4q}{n^3}$.

Then $y_1 = n \cosh A$,

$2y_2 = -y_1 + iv\sqrt{3}$, and

$2y_3 = -y_1 - iv\sqrt{3}$ where $v = n \sinh A$.

III. p is positive.

Set $n^2 = \dfrac{4p}{3}$ and $\sinh 3A = \dfrac{-4q}{n^3}$.

Then $y_1 = n \sinh A$,

$2y_2 = -y_1 + iv\sqrt{3}$, and

$2y_3 = -y_1 - iv\sqrt{3}$ where $v = n \cosh A$.

Functional Equations Defining the Complementary Operations*

By S. T. SANDERS, Jr.
Nebraska State Teachers' College, Chadron, Neb.

Though there is a considerable body of mathematical literature devoted to the subject of functional equations, the type of equation involving functions whose arguments and whose values are sets of points seems to have received little attention. In particular, the characterization of set operations by means of such a functional equation has escaped consideration. In this paper certain equations are studied which serve to define the complementary operation.

For every set of points, A, in space, S, it is assumed that the set fA is uniquely determined; likewise, the inverse set $f^{-1}A$. We consider the following five characterisitic properties of the complement, cA, of A, and proceed to determine those combinations of properties which are sufficient to define the complement.

I. $$A+fA=S.$$

II. $$AfA=0.$$

III. $$f(A+B)=fAfB.$$

IV. $$f(AB)=fA+fB.$$

V. $$f^2A=A.$$

Properties I and II define at once the complement. For by I we have

$$AcA+fAcA=ScA;$$

while by II, $$fA\subset cA,$$

so that the above equality becomes

$$fA=cA.$$

Properties I and III likewise define the complement. From III follows readily the monotonic property,

(1) $$A\subset B\rightarrow fB\subset fA.$$

*Presented to the Nebraska section of the Mathematical Association of America, May 7, 1937, Lincoln, Neb.

From this, by the uniqueness of definition of $f^{-1}A$ for every set A, we have

$$fS=0.$$

Thus, setting $B=cA$ in III,

$$fAfcA=fS=0,$$

whence

(2) $$cfA+cfcA=c0=S.$$

Now by I,

(3) $$cfA\subset A,$$

and

(4) $$cfcA\subset cA,$$

whence

(5) $$cfAcfcA=0.$$

It is evident that (2) and (5) have the form of properties I and II, so that we have

$$cfA=fcA,$$

by which (4) becomes

$$fA\subset cA.$$

The reversed inclusion is obtained from (3).

Properties I and IV imply II, and so define the complement. Setting $B=fA$ in IV, we obtain by I,

$$f(AfA)=fA+f^2A=S;$$

but by I, $f0=S$. Hence by unique inverses,

$$AfA=0.$$

Properties I and V are not sufficient to define the complement, as the following example shows. Consider the linear space of all integers, where we define $fA=cA$, unless A is the set of all integers not less (or not more) than n, in which case we define $fA=cA+n$.

The property combination, II-III defines the complement, the demonstration being similar to that of the case I-IV. Likewise, the procedure for the combination II-IV follows the lines of that of I-III. Similarly an example may be found showing that II and V are not sufficient to characterize uniquely the complement. These analogies

might be expected from the symmetry of the property pairs, I-II and and III-IV with respect to the operations, product and sum.

Properties III and IV are far from sufficient to define the complement. Indeed, III, IV, and V will not serve the purpose, as is seen by a space of two points, a_1, a_2, where fA is defined as follows:

$$fa_1 = a_1;\, fa_2 = a_2;\, fS = 0;\, f0 = S.$$

In summary, the complementary operation is defined by any pair of the functional equations, I-IV, with the exception of the pair III-IV.

Garrett's Mechanism

By W. C. JANES
Kansas State College of Agriculture and Applied Science

In the February, 1930, issue of the *American Mathematical Monthly* Professor W. H. Garrett has discussed *The Problem of Two Bodies and a Related Sextic*. Near the end of the article Professor Garrett describes a mechanical method of drawing the sextic, but he does not mention the fact that his mechanism will draw a somewhat more general sextic than discussed in his paper.

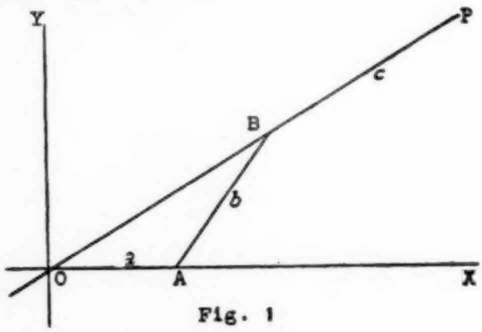


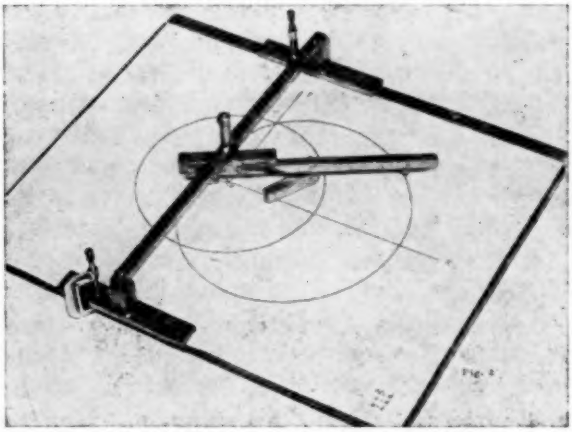
Fig. 1

Suppose in Fig. 1 that the lengths of OA, AB, and BP are respectively a, b, and c. Let A represent a fixed pivot about which the crank of length b may rotate. Let the crank AB be pivoted to the movable rod OP at B. Let O represent a pivot such that OP may both slide and rotate at O. The problem is to determine the locus of P.

The equation of the required locus is

$$\begin{aligned}(x^2+y^2)^3-4ax(x^2+y^2)^2+2(a^2-b^2-c^2)(x^2+y^2)^2 \\ +4a^2x^2(x^2+y^2)-4ax(a^2-b^2-c^2)(x^2+y^2) \\ +(a^2+c^2-b^2)^2(x^2+y^2)-4a^2c^2x^2=0.\end{aligned}$$

The equation shows the locus to be a sixth degree curve, symmetrical with respect to the x-axis, and with the origin as a double point. P is shown in Fig. 1 at a distance c from B on the opposite side of O; but as the equation involves only even powers of c, the same result would be obtained if P were chosen at a distance c from B on the same side as O. Hence the locus can be described if two pencils are placed on the bar OP at distances c on opposite sides of B. This shows that the curve consists of two branches.



The x-intercepts are

$$x=0,\ 0,\ a+b+c,\ a+b-c,\ a-b+c,\ a-b-c.$$

An interesting situation arises when $b^2=(c\pm a)^2$. In this case the curve has a cusp at the origin.

An important special case also arises when $b=a$. If $b=a$ the left member of the equation factors, and the locus degenerates into a circle and a limacon. The equation then becomes

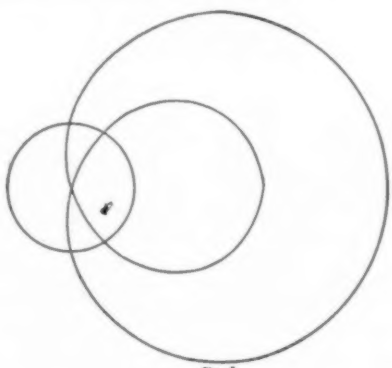
$$(x^2+y^2-c^2)[(x^2+y^2-2ax)^2-c^2(x^2+y^2)]=0.$$

It is seen that when

$c>2a$ the limacon has an isolated interior point,

$c=2a$ the limacon is a cardioid,

$c<2a$ the limacon has an interior loop.



Consequently the mechanism can be made to draw any desired type of limacon (Garrett mentions only the cardioid). However, in using the instrument for the purpose of drawing limacons a difficulty arises due to the fact that the motion becomes indeterminate at the point $(0,c)$ where the limacon crosses the circle.

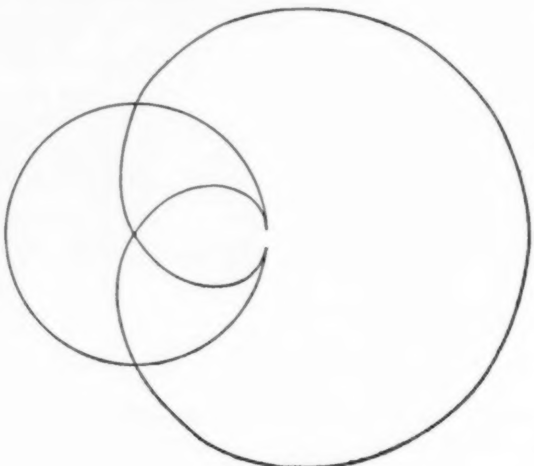


Fig. [illegible]

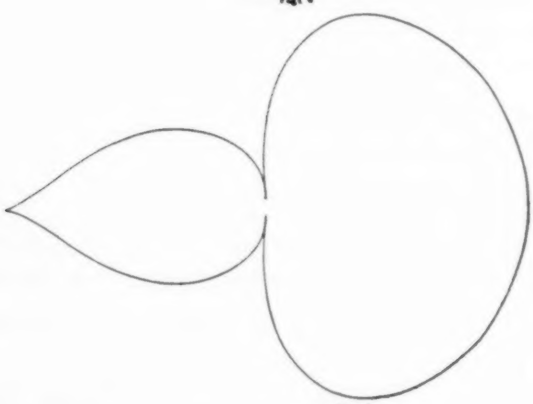


Fig. [illegible]

The general equation which defines the locus of P represents a three-parameter family of curves. But all of the types of curves defined by this equation can be obtained by studying only a two-parameter family. Suppose, for instance, that the curve is drawn for the case $a=2$, $b=3$, $c=4$. Suppose further that the curve for the case $a=6$, $b=9$, $c=12$ is also drawn. It is seen that the values assigned in the second set are proportional to those of the first set. A glance at the nature of the mechanism shows that the second curve differs

from the first only in the fact that it is drawn to a scale three times as large. A convenient type of design for the mechanism results when a fixed value is assigned to the length of the crank b. Then for a fixed value of c there results a single infinitude of curves by allowing a to vary. There will furthermore be a single infinitude of such families of curves if the length of the arm c is allowed to vary. The three parameters a, b, and c may have any real values. However, it is of little importance to discuss the cases when any of them are negative. If a is negative, the resulting curve is merely a reflection with respect to the y-axis of the curve obtained when a is positive. Mounting the two pencils on opposite sides of B takes care of both positive and negative values of c, since one branch of the curve is the locus obtained for a positive c and the other branch is obtained with a negative c.

It is interesting to notice the change in shape of these branches as the parameters are allowed to vary. As a sort of guide to the nature of the concavity of a branch it is well to note that when

$a=\sqrt{bc}-b$, the curvature at the point $(\sqrt{bc}-c,\ 0)$ is zero. If $a=\sqrt{bc}+b$, the curvature at the point $(\sqrt{bc}+c,\ 0)$ is zero.

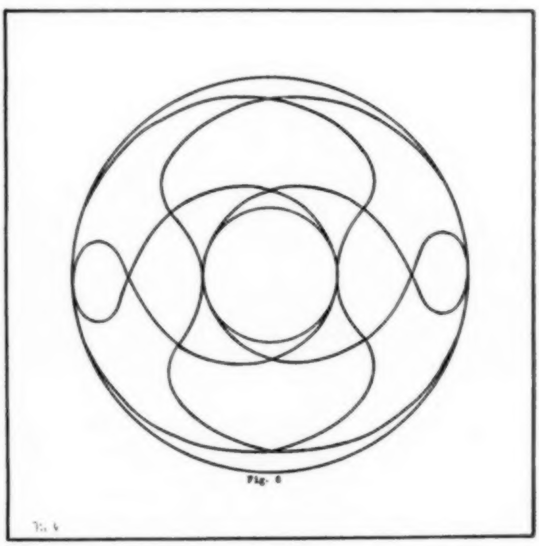
Fig. 2 is a picture of a crudely constructed mechanism for drawing the sextic.

Fig. 3 shows the curve for $a=2$, $b=2$, $c=1$.

Fig. 4 shows the curve for $a=2$, $b=2$, $c=2$.

Fig. 5 shows the curve for $a=4$, $b=2$, $c=2$.

Fig. 6 indicates what may result if several curves are drawn on the same sheet of paper.



Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON

Report of the Euler Commission

By ANDREAS SPEISER
University of Zurich, Switzerland

(EDITOR'S NOTE.—We have been favored with the following fine report from Professor Speiser, a member of the Euler Commission, on the present status of its work, but for American readers deem it necessary to give something of its historical background, as follows:

The Swiss Society of Natural Sciences (Schweizerische Naturforschende Gesellschaft) at its annual meeting in Glarus on August 31, 1908, created the Euler Commission to edit the Collected Works of Leonhard Euler, in conjunction with the Central Committee. On September 6, 1909, the same Society decided on "the edition of the Collected Works of Leonhard Euler in the original languages, convinced of rendering the entire scientific world a service thereby." In the 1840s C. G. J. Jacobi had planned to edit the works of Euler, but the plan did not materialize; he died several years later. The late Prof. Paul Stäckel, indefatigable as always, performed the laborious task of setting up an index of the published papers of Euler; he arranged the mass of material according to subject matter and indicated the planned contents for the separate volumes. He was one of the principal early editors.

Stäckel used earlier Euler indices by P. H. von Fuss (1843), G. Hagen (1896) and F. Müller (1908) as well as Jacobi's letter to Fuss dated March-April, 1848. The separate papers have been arranged largely in chronological order, because the date of publication rather than of composition could be definitely determined in each case.

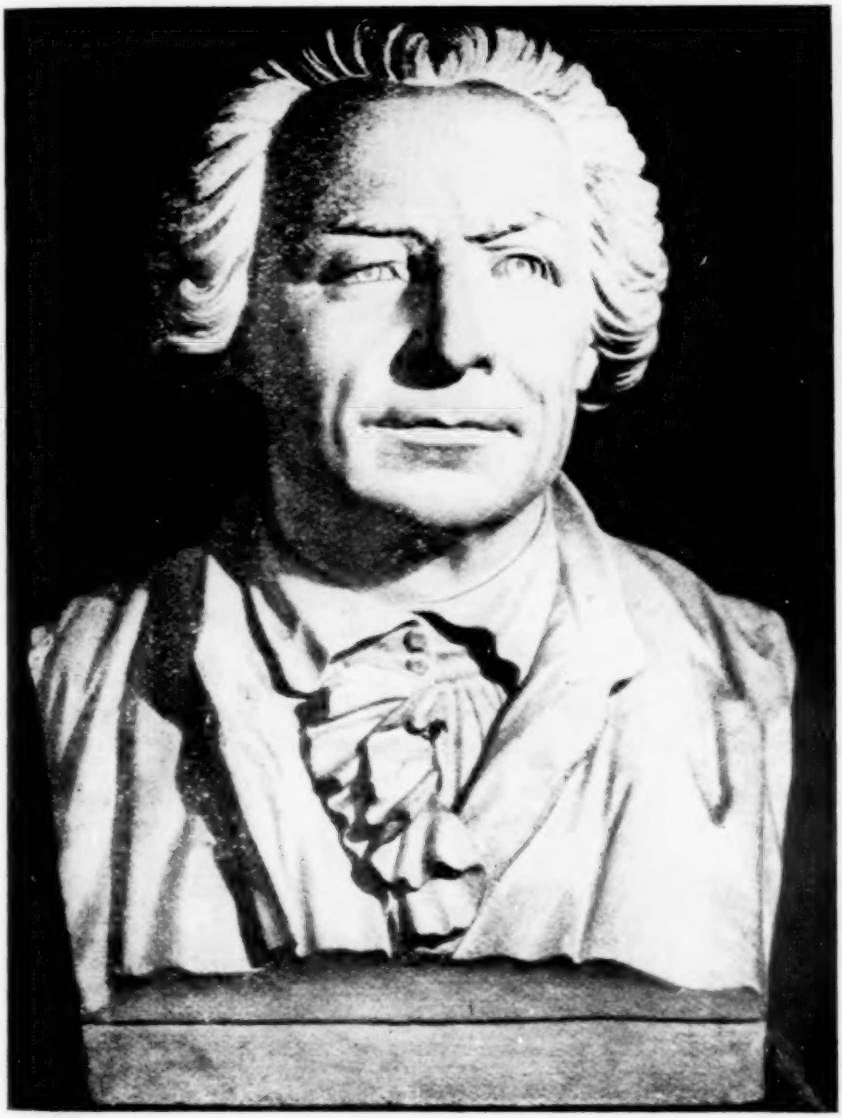
The Deutsche Mathematiker-Vereinigung at its annual meeting in Cologne on September 22, 1909, voted a contribution of 5000 francs to the edition and pledged its members to scientific coöperation in the task of editorial research. The Swiss Society succeeded in gaining the interest of the entire scientific world in the project; 350 subscribers for the entire edition were quickly enrolled and in addition 135,000 francs in cash were collected. For many years the late Prof. Ferdinand Rudio (Zurich) served as general editor. The volumes appeared with the imprimatur of the firm B. G. Teubner in Berlin and Leipzig. In 1910 Teubner published G. Eneström's excellent index of Euler's works. The Russian Academy in Leningrad opened up all the manuscripts in its archive to the committee.

Jacobi had thought of including the scientific papers of Euler's son Johann Albrecht in the edition. However, it was soon found that these needed considerable corrections and additions. Stäckel (and Eneström) published a bibliography and brief sketch of Johann Albrecht's life. His papers are concerned with applied rather than pure mathematics.

Euler published his papers in Latin, French, German and several in English. It is expected that his *Letters to a German Princess* will occupy one of the quarto volumes, in Series III, and the correspondence with d'Alembert, the Bernoullis, Goldbach, Lagrange and others will fill at least three volumes—G. W. D.)

* * * *

The Collected Works of Leonhard Euler will embrace approximately seventy volumes. From 1911 to 1937 twenty-six of these volumes have appeared, of which twenty volumes belong to Series I



Heinrich Ruf's Marble Bust of
LEONHARD EULER
in the Bernoullianum at Basel
Switzerland
1875

which contains the mathematical works. Series II contains the works on mechanics, technology and astronomy, while Series III is devoted to the works on physics and philosophy.

In 1911 at the beginning of the undertaking the work could be regarded as well founded financially, thanks to the many subscribers and the large sum of money which was collected by the Euler Commission. Almost half of the subscribers were lost through the World War, furthermore the costs of printing rose to such an extent that we were compelled essentially to slow up the tempo of the edition.

Among our patrons those of the U. S. A. assume a distinguished position. There we lost no subscribers and we were supported by the American Mathematical Society with considerable financial contributions.

I should like to take the liberty of reporting on the volumes which have appeared most recently. The editing of the papers on series was completed a short while ago in volumes 14-16 of Series I. An exhaustive preface by Professor G. Faber permits an orientation in the enormous amount of material which is contained here. It is shown that even today only a small part is worked over or digested and that much stimulation will proceed from it. Above all one recognizes that Euler by no means worked naively with divergent series, but that he used them with full consciousness as an inventive method. With them he was successful in a series of the most important discoveries which could scarcely have been made in another way, e. g., the so-called Riemann functional equation of the zeta function. The view that one may employ only convergent series was regnant even in the 18th century and Euler combats it with the reference to the power of divergent series. Therefore Euler's conception was bold for that time and he justifies it by his success.

Thus I should like to emphasize, above all, that the Eulerian works have not only an historical interest, but that today they still form a rich store-house for researches.

Two volumes on the theory of numbers (ed. by Professor R. Fueter, Zurich) are still in preparation, furthermore two on differential equations (ed. by Professor Dulac in Lyon) of which the first has already appeared, then two volumes on the calculus of variations (ed. by Professor C. Carathéodory in Munich). Then the works on technology will be edited (by Professor Ackeret in Zurich), which will deserve the special interest of engineers. Euler is the real inventor of the turbine and his calculations are serviceable even today. We hope especially to obtain subsidies from directorates of electrical utilities, for they profit preëminently by the Eulerian calculations.

In closing I should like to refer to the correspondence of Euler. We already have a large number of copies and photostats, but many are not yet found. It is almost certain that they still exist today, but probably are in the hands of private collectors or in archives whose existence is not known to us. It would be of the utmost importance for us, if the names of such letter owners were communicated to us, so that we could get in touch with them. Before we begin with the publication of this material we should have a survey of that which is extant.

The History of Mathematics in Hungary Before 1830*

By DR. JÓZSEF JELITAI
Budapest, Hungary

Translated by Departmental Editor
G. Waldo Dunnington

The first printed mathematical work of a Hungarian author is a Latin incunabulum *Arithmetice summa tripartita Magistri georgij de hungaria,* dated 1499. This work with 20 pages, vigorous Gothic type, and many abbreviations originated supposedly in a Dutch printing-shop. It is a well-rounded presentation of contemporary arithmetic. Boetius and Brawardinus are mentioned. Peculiar numerals occur, which are otherwise found only in Pedro Sanchez Ciruelo. The passage "aureas ytalorum hungarorumque regulas" refers perhaps to a contemporary, specifically Hungarian process of calculation. The municipal library of Hamburg and the Hungarica Collection of Count Alexander Apponyi (now in the Hungarian National Museum in Budapest) each possess a copy.

The first (and indeed) practical geometry from a Hungarian pen is a stately German book of 258 pages with many beautiful illustrations: *Ein kurtze und grundliche analytung zu dem rechten verstand Geometriae.* Durch Christoffen Puehler von Syclas (Siklós) in Ungern. It was printed in Dillengen, Bavaria, in 1563 and treats the contemporary instruments and methods of surveyors and astronomers.

The 150 years of Turkish sovereignty in Hungary make it understandable why the first mathematical work written in Hungarian, consisting of 140 pages by an unknown author, did not appear until 1577. The only copy of this arithmetic of Debrecen is in the Széchenyi provincial library of the Hungarian National Museum in Budapest. The second edition of 1582, also in Debrecen, remained unchanged, but the third edition of 1591, Kolozsvár-Klausenburg, was considerably improved and enlarged to 232 pages.

The eleventh "Liber Regius" of Prince Gabriel Bethlen of Transylvania, now in the Budapest provincial archive, contains on the first 45 folio pages a geometry with many figures and arithmetical examples

*Lecture before the International History of Science Congress, Prague, September 23, 1937.

in Old French. The manuscript begins with the "Principes de Geometrie", there follow "Definitions, Operations, des Angles, des Cercles", constructions e. g. of "Heptagone, Nonagone, Undecagone, Ovale", "De Changement des Figures" and at the close "De la mesure des superficies". The text, style, water-mark of the paper and the residual content of the book seem to indicate the period around 1600.

The first encyclopaedia written in Hungarian (1653), by J. Apácai Csere (1625-1659) includes also arithmetic and geometry. The treatment of these parts however is not independent and does not reach the higher level of that time. The Hungarian language is indebted to Apácai for some excellent mathematical nomenclature.

In the years 1743, 1763, and 1782 appeared the unchanged editions of the excellent Hungarian arithmetic by Georg Maróthi (1715-1744), professor at the College of Debrecen. He was such a clear-sighted educator that with him number and form—indeed 60 years *before* Pestalozzi—belong to the elementary instructional aids. His memory lives even today in the winged word: It's also thus in Maróthi!

Stefan v.Hatvani (1718-1786), professor at the College of Debrecen, was the first in Hungary to work in the philosophy of mathematics and with the calculus of probability. The clear expositions of his work *Introductio ad principia philosophiae solidoris*, Debrecen, 1757, are in every respect at the high level of his time. Hatvani was a gifted and versatile pupil of Daniel and Johann II Bernoulli.

Paul Makó de Kerekgede (1724-1793) was active as dean of the philosophical faculty at the Pázmány University. He acquired great merit as joint author of the *Ratio Educationis*. His mathematical textbooks, which appeared in Vienna, found wide circulation in Hungary as well as in foreign countries.

Paul Sipos (1759-1816) was a distinguished pupil of the Bethlen College at Nagyenyed in Transylvania. His teacher in mathematics was Josef v. Kováts (1732-1795), one of the most excellent professors of this famous school. Imperial Count Samuel Teleki, the subsequent Transylvania Court Chancellor in Vienna, took this Kováts as a companion on his great study tour of 1759-1763 to Basel, Utrecht, Leyden and Paris. From the Hungarian diaries of the Count it is evident that Johann II Bernoulli gave them both in Basel day by day private instruction in higher mathematics. There they also studied for two months later mechanics under Daniel Bernoulli.

Sipos was later active as a tutor in the family of the Keeper of the Crown, Count Josef Teleki. The latter also received during his studies in Basel for nine months private instruction from Daniel Bernoulli. Both Telekis attended the public lectures of Daniel Ber-

noulli on experimental physics and the private lectures of A. Socin on electricity. Since reports of students concerning such courses of that period are scarcely known, it is a special stroke of good fortune that Josef Teleki in his diary written in Hungarian describes several experiments (September, 1759): The German translation of these passages is found in the book by O. Spiess: ***Basel anno 1760. Nach den Tagebüchern der ungarischen Grafen Joseph und Samuel Teleki.*** (Basel, 1936.)

In Paris the Counts Teleki associated primarily with Clairaut, La Condamine, d'Alembert, La Caille, Lalande, Montucla, Fontaine, Nollet and Rousselot. Well known in the history of the calculus of probability is the essay of Daniel Bernoulli on vaccination and the polemic of D'Alembert which the latter delivered in the Academy of Sciences, on which occasion Josef Teleki was present. I owe the the following communication to the kindness of O. Spiess:

"Since Daniel hated polemics, Teleki undertook his defense by composing a long document (now in the Gotha Collection, in Basel), in which he attacks d'Alembert rather sharply. Unfortunately the conclusion of this work is missing." O. Spiess also identified a notebook of Daniel Teleki. It is a Latin version of a private course under Daniel Bernoulli on mechanics, which closes with the theory of vital forces. There are about 360 pages, rather closely written, especially carefully and adorned with many figures; the many practical examples are characteristic.

The first Latin paper of Sipos (1792) treats in 199 paragraphs the conic sections. Definitely new material is contained in the appendix: ***De curva accentrica.*** In a later Sipos Manuscript (1793) this new curve is called: ***Quadratix Spiralis.*** This transcendental Sipos curve, the cochleoid, renders possible the solution of twelve problems which Sipos set himself, as e. g. the measurement of the length of a circular arc, division of an arc or angle division. The method of Sipos yields according to my calculations the following approximation formula for the circumference of an ellipse (semi-axes: a, b):

$$\frac{4(a+b)^2}{(a-b)^2}\cos\left(\frac{\pi\sqrt{ab}}{a+b}\right)$$

Maximum error 1.4%, when $b=0.1a$

My historical monograph of 1932 (39 approximation formulas for the circumference, 22 of these developed in series) proves the excellence of this formula. According to my calculations it is superior to both—even trigonometrical—approximations of Soreau (1913) and is even

today one of the best approximations. In 1795 Sipos received a golden medal from the Berlin Academy. Between 1700 and 1854 he was the the only mathematician whose work the Berlin Academy published in its proceedings, (1790-91, pp. 201-30) without having been a member.

The logarithmic-trigonometrical tables of Sipos appeared in 1807. They are the first of Hungary with decimal angle division. As shown by my series developments, the logarithm of a trigonometrical function $f(x)$, aside from a member $a \log x$, contains only even powers of x:

$$\log f(x) = a \log x + bP(x^2)$$

where a and b represent constants. This equation explains the construction of the Sipos Tables, which, although their structure is peculiar, have not been considered in mathematical literature.

Sipos taught from 1805 to 1810 at the famous College at Sárospatak. The excellent mathematical part of the curriculum of this school of 1810 is probably due to him.

As a philosopher and pulpit orator Sipos was first under the influence of Kant, whose thought-world he grasped better than anyone else of his day in Hungary; later he represents the subjective idealism of Fichte. The leading figure in contemporary Hungarian literature, Franz v. Kazinczy counted him among his best friends. Sipos was also gifted as a poet, his Latin and Hungarian poems were highly esteemed by Kazinczy.

For at least eleven years there studied in Debrecen Ladislaus Chernac Pannonius (b. 1740 in Pápa – d. 1816), later professor at the Athenaeum in Deventer, Holland. His chief work is the *Cribrum arithmeticum*, Daventriae, 1811, "a large very meritorious work of tables, which historically deserves so much more attention, since it is to a high degree reliable. In the Chernac work are indicated probably for the first time all prime numbers lying between the given limits, up to 1020000." (F. Klein). These tables of factors, model in construction and execution, were especially praised by Gauss, Legendre, Delambre and Crelle.

The University of Budapest was founded by Cardinal Pázmány as a Jesuit university in Nagyszombat (Tyrnaviae). After the dissolution of the order it was moved as the Royal Hungarian University in 1777 to Buda and in 1784 to Pest. Even in 1774 the great queen, Maria Theresa, wanted to found a normal school, i. e. "a seminary of teachers according to the example of the dissolved society". The Royal Hungarian-Aulic Chancellery proposes for the "Professor Repetentium Matheseos" an annual salary of 1200 florins. The first

"Professor Matheseos Sublimioris per Concursum constitutus" was the clergyman Josef v. Mitterpacher; his colleagues: the Professor matheseos elementaris pro logicis, Andreas Dugonics and after 1777 also a third mathematician, the Professor matheseos practicae: Franz Rausch. Even in 1782 there receive six Repetentes Ordinarii Stipendiati "ex Mathesi Sublimiore: primam Eminenter". We read in the printed catalogue of courses for the academic year 1785-86: "Mathesis sublimior pura, in cursum triennalem distributa illis, qui Philosophiam absolverunt, diebus Lunae, Mercurii, Veneris & Sabathi a 10ma usque 11mam praelegitur secundum institutiones Scherfferi". J. Pasquich was a pupil of Mitterpacher, whose algebra and analysis appeared in the first volume of his "Instruction". Pasquich dedicated the book to King Leopold II, who conferred on him a gold chain for it. Emperor Josef supplemented in 1782 the philosophical faculty with an Institutum Geometrico-Hydrotechnico-Practicum, the germ of today's Budapest Institute of Technology. A class index of the year 1785 lists 52 geometers who in three classes were studying hydraulics, practical drawing, engineering and agriculture. For these students prizes were distributed semi-annually, totaling 400 florins. An inventory of 1786 shows 41 teaching models with a total value of 1832 florins and 25 kreuzers.

Pasquich resigned his professorship in 1797. The minutes of the faculty on this occasion fill in the archive 24 sheets. The faculty proudly refers to the fact that the late Emperor Josef in his severe reform did not find any fault with the instruction in higher mathematics. Of the six proposals presented the plan of the adjunct astronomer Bruna was unanimously accepted. He enumerates the material of instruction on five folio-half-sheets, with 63 references to the literature.

As director of the Ofen Observatory, Pasquich later wanted to have a pupil of Gauss as associate. I found in the Budapest Provincial Archive a letter of Gauss dated July 30, 1814, and parts of two earlier letters by him, where he recommends Encke and Gerling for this position. In a hitherto unpublished letter Encke asked Pasquich for an extension of time.

In 1830 appears the first mathematical work of Wolfgang Bolyai, with his son John the greatest double star of the mathematical firmament in Hungary.

Further research must determine the validity and possible gaps in this first sketch. I owe sincere thanks to Prof. Lajos v. David (Debrecen), who aroused in me an interest in mathematico-historical investigations.

EDITOR'S NOTE

The reader who is interested in the material presented above by Dr. Jelitai will find extremely valuable his following works:

1. *Sipos Pál Élete és Matematikai munkássága,* írta Woyciechowsky József, Athenaeum, Irodalmi és Nyomdai Részvénytársulat Nyomása, Budapest, 1932. (124 pages, with illustrations, edited by Prof. L. v. David, in the series Közlemények a Debreceni Tudományegyetem Matematikai Szemináriumából, VI. Füzet; contains also hitherto unpublished letters of Bode and Kästner to Sipos.)

2. *Sipos Pál egy Kézirata és a Kochleoid,* írta Woyciechowsky József, Matematikai és Fizikai Lapok, XLI kötet 1 füzeteből, Budapest, 1934. (p. 45-54).

3. *Sipos-Kéziratok a Gyömröi Teleki-Levéltárban,* írta Woyciechowsky József, Matematikai és Fizikai Lapok, XLII kötet 2 füzeteből, Budapest, 1935. (p. 134-138).

4. *Bernoulli Dániel és János egykorú Teleki-útinaplók és levelek tükrében,* írta J. Jelitai, [in the same periodical as No. 2 and No. 3, p. 142-160, Vol. XLIII, No. 2.] Budapest, 1936.

5. *Zur Geschichte der Mathematik in Ungarn,* J. v. Jelitai, Comptes Rendus du Congrès International des Mathématiciens, Tome II, Conférences de Sections, Oslo, 1937. (p. 279.) G. W. D.

The Teachers' Department

Edited by
JOSEPH SEIDLIN and JAMES McGIFFERT

The Introduction of Invariant Theory Into Elementary Analytic Geometry

By L. E. BUSH
The College of St. Thomas

PART II. SECOND DEGREE SURFACES

7. *Invariants and canonical forms.* In rectangular coordinates, the equation

(11) $ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2lx+2my+2nz+d=0,$

where the coefficients are all real and at least one of those of the second degree terms is not zero, represents a *second degree surface.* The functions

$$I=a+b+c,\ J=bc+ca+ab-f^2-g^2-h^2,$$

$$D=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix},\quad \Delta=\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & d \end{vmatrix}$$

are Euclidean invariants of (11).* They are of weights 1, 2, 3, and 4 respectively.

Every second degree surface can be reduced by a proper choice of axes of reference to one of the following forms:†

(12) $ax^2+by^2+cz^2-1=0,\ abc\neq 0,$

(13) $ax^2+by^2+cz^2=0,\ abc\neq 0,$

*V. Snyder and C. H. Sisam, *Analytic Geometry of Space*, New York 1914, pp. 74-86. The invariants are here obtained by rather intuitive processes and their invariance then established.

†Snyder and Sisam, loc. cit., pp. 74-87. Although the forms of the equations as given are not identical with those given by Snyder and Sisam, they can in each case be obtained by multiplication by some constant.

(14) $ax^2+by^2+2z=0,\ ab\neq 0,$

(15) $ax^2+by^2=0,\ ab\neq 0,$

(16) $ax^2+by^2-1=0,\ ab\neq 0,$

(17) $y^2+2lx=0,\ l\neq 0,$

(18) $y^2+d=0,\ d\neq 0,$

(19) $y^2=0.$

The invariants have the values given below for these respective canonical forms:

(12) $I=a+b+c,\ J=bc+ac+ab,\ D=abc,\ \Delta=-abc,$

(13) $I=a+b+c,\ J=bc+ac+ab,\ D=abc,\ \Delta=0,$

(14) $I=a+b,\ J=ab,\ D=0,\ \Delta=-ab,$

(15) $I=a+b,\ J=ab,\ D=\Delta=0,$

(16) $I=a+b,\ J=ab,\ D=\Delta=0,$

(17) $I=1,\ J=D=\Delta=0,$

(18) $I=1,\ J=D=\Delta=0,$

(19) $I=1,\ J=D=\Delta=0.$

These values show that it is impossible by means of the invariants at hand to distinguish between surfaces which reduce to (15) and those which reduce to (16), nor can we distinguish between those which reduce to (17), those which reduce to (18), and those which reduce to (19). We must therefore seek other functions which will serve to make these distinctions.

We note that (15) represents a pair of planes and (16) a cylinder. We therefore ask for the condition that (11) represent a pair of planes. An examination of the canonical forms shows that whenever (11) represents a pair of planes, $D=\Delta=0$, but that this is not sufficient, since it holds also for cylinders. Let us consider first the simple case when $c=f=g=n=0$ in (11). This is either a cylinder with its elements perpendicular to the XY-plane or a pair of planes perpendicular to the XY-plane. For it to be the latter, its trace on the XY-plane must be a pair of lines. Hence, by the results obtained in Part I, § 4,

(20) $(ab-h^2)d-bl^2-am^2+2hlm$

must vanish. It is obvious, however, that this function is not an

invariant of (11), since the simple rotations of axes $x=y'$, $y=z'$, $z=x'$ and $x=z'$, $y=x'$, $z=y'$ change it into

(21) $(ac-g^2)d-an^2-cl^2+2gln$

and

(22) $(bc-f^2)d-cm^2-bn^2+2fmn$

respectively. In fact, these two rotations show that an invariant of (11) must be unaltered whenever the same cyclic permutation is made in each of the three sets of coefficients, (a, b, c), (f, g, h), and (l, m, n). Furthermore, it is seen that the vanishing of (21) and (22) are the conditions for (11) to represent a pair of planes in the respective cases $b=h=f=m=0$ and $a=g=h=l=0$. The simplest function which reduces to (20), (21), and (22) when $c=f=g=n=0$, $b=h=f=m=0$, and $a=g=h=l=0$ respectively, and which is unaltered by the suggested permutations, is the sum of (20), (21), and (22), namely

$$S=Jd-(b+c)l^2-(a+c)m^2-(a+b)n^2+2hlm+2fmn+2gln.$$

We find upon testing S that it is invariant under rotation,* but not invariant under translation. If the origin be translated to (x_0, y_0, z_0), we find that

$$S'=D(x_0^2+y_0^2+z_0^2)-2Lx_0-2My_0-2Nz_0+S,$$

where L, M, and N are the cofactors of l, m, and n in Δ. But suppose $D=\Delta=0$. Let $\bar{a}$, $\bar{h}$, and $\bar{g}$ be the cofactors of a, h, and g in D. Multiply the first row of Δ by $\bar{a}$ and add to it the elements of the second and third rows multiplied by $\bar{h}$ and $\bar{g}$ respectively. Expanding now by the elements of the first row, we obtain $\bar{a}\Delta=L^2=0$. Similarly $M=N=0$. Thus $S'=S$, and S is an invariant of (11) of weight 3 whenever $D=\Delta=0$.

For (15), $S=0$; for (16), $S=-ab$; for (17), $S=-l^2$; for (18) and (19), $S=0$. Thus we are not yet able to distinguish between surfaces which reduce to (18) and those which reduce to (19). These two forms represent non-coincident parallel and coincident planes respectively, and for both $S=J=D=\Delta=0$. We ask for the further condition that the planes be coincident. Again, we first consider the simple case where $c=f=g=n=0$ in (11), and find the condition to be l^2+m^2-ad

*In testing for invariance under rotation the computation can be reduced enormously by considering the total rotation as composed of three successive rotations, in each of which one axis is held fixed. See Snyder and Sisam, loc. cit., p. 42. It is evident that if the function is unaltered by the permutations suggested above and is also unaltered by every rotation in which one particular axis is held fixed, then it is unaltered by every rotation of axes.

$-bd=0$. Reasoning similar to that used above leads us to consider the function

$$T=l^2+m^2+n^2-Id.$$

This function is found to be invariant under rotation of the axes, but when the origin is translated to (x_0, y_0, z_0), it becomes

$$\begin{aligned} T' = &-(ab-h^2+ac-g^2)x_0^2-(ab-h^2+bc-f^2)y_0^2-(ac-g^2+bc-f^2)z_0^2 \\ &+2(hg-af)y_0z_0+2(hf-bg)z_0x_0+2(fg-ch)x_0y_0 \\ &+2(hm-bl+gn-cl)x_0+2(hl-am+fn-cm)y_0 \\ &+2(gl-an+fm-bn)z_0+T. \end{aligned}$$

Thus T is not invariant under translation of the axes. But, if $S=J=D=\Delta=0$, (11) is a pair of parallel (or coincident) lines and can therefore be written

$$(\alpha x+\beta y+\gamma z+\delta)(\alpha x+\beta y+\gamma z+\delta')=0,$$

where $a=\alpha^2$, $b=\beta^2$, $c=\gamma^2$, $f=\beta\gamma$, $g=\alpha\gamma$, $\mathrm{H}=\alpha\beta$, $l=\frac{1}{2}\alpha(\delta+\delta')$, $m=\frac{1}{2}\beta(\delta+\delta')$, $n=\frac{1}{2}\gamma(\delta+\delta')$. Making use of these values in the expression for T', we get $T'=T$. Therefore T is an invariant of (11) of weight 2 whenever $S=J=D=\Delta=0$.

8. *Parameters in terms of invariants.* We note that (13) and (15) are homogeneous in the coefficients, hence their coefficients are invariants of weight one. The coefficients of each of the other canonical forms are geometric invariants. We shall express all of these coefficients in terms of I, J, D, Δ, S (in case $D=\Delta=0$), and T (in case $S=J=D=\Delta=0$). The following values can be verified with ease.

For (12), a, b, and c are the three roots (in any order) of the cubic equation $\Delta^3x^3+ID\Delta^2x^2+JD^2\Delta x+D^4=0$. It is obvious that, since the coefficients of this equation are all invariants of the same weight (i. e., twelve), its roots are geometric invariants.

For (13), a, b, and c are the three roots of the cubic $x^3-Ix^2+Jx-D=0$. Since I, J, and D are invariants of weights one, two, and three respectively, the roots of this equation are invariants of weight one. Since $D\neq 0$, none of the roots is zero. Hence we may divide out any one of the coefficients of (13), the two remaining coefficients then being geometric invariants, namely the quotients of two of the roots of the above cubic by the third root.

For (14), a and b are the two roots of the quadratic $\Delta x^2+I(-J\Delta)^{1/2}x-J^2=0$. Since the coefficients are all of the same weight (four), the roots of this equation are geometric invariants.

For (15), a and b are the two roots of the quadratic $x^2-Ix+J=0$. As in (13), the roots of this equation are invariants of weight one. The equation may be divided by either a or b, leading to a canonical form with one geometrically invariant parameter.

For (16), $D=\Delta=0$ and S is invariant. We find that a and b are the roots of $S^2x^2+IJSx+J^3=0$.

For (17), $D=\Delta=0$ and S is invariant. We may use for l either root of $I^3x^2+S=0$.

For (18), $S=J=D=\Delta=0$ and T is invariant. Here $d=-T/I^2$.

9. *Classification of second degree surfaces.* In (12) and (13) it is evident that both $ID>0$ and $J>0$ whenever a, b, and c all have the same sign. Conversely, if $ID>0$ and $J>0$, the coefficients a, b, and c will all be of like sign.* An examination of the canonical forms and the invariants leads to the following classification.

I. $\Delta\neq0$, $D\neq0$, ellipsoid or hyperboloid.
- (a) If $ID>0$ and $J>0$, an ellipsoid, which is real or imaginary according as $\Delta<0$ or $\Delta>0$.
- (b) If either $ID\leq0$ or $J\leq0$†, an hyperboloid, which is of one or two sheets according as $\Delta>0$ or $\Delta<0$.

II. $\Delta=0$, $D\neq0$, a quadric cone, which is imaginary if both ID and J are positive, otherwise real.

III. $\Delta\neq0$, $D=0$, paraboloid.
- (a) If $\Delta<0$, it is an elliptic paraboloid.
- (b) If $\Delta>0$, it is an hyperbolic paraboloid.

IV. $\Delta=D=S=0$, $J\neq0$, a pair of intersecting planes, which are real or imaginary according as $J<0$ or $J>0$.

V. $\Delta=D=0$, $JS\neq0$, an axial cylinder.
- (a) If $J>0$, an elliptic cylinder, which is real or imaginary according as $IS<0$ or $IS>0$.
- (b) If $J<0$, an hyperbolic cylinder.

VI. $\Delta=D=J=0$, $S\neq0$, a parabolic cylinder.

*This can be proved directly from the formulas for I, J, and D. But it is more easily proved by noting that in both (12) and (13) a, b, and c are proportional to the roots of $x^3-Ix^2+Jx-D=0$. Application of Descartes' Rule of Signs to this equation leads to the desired result.

†It is interesting to note that, since a, b, and c are real and different from zero, if ID and J are not both positive, then at least one of them is negative. This can be proved by the method of the last footnote.

VII. $\Delta = D = J = S = 0$, $T \neq 0$, parallel non-coincident planes, which are real or imaginary according as $T > 0$ or $T < 0$.

VIII. $\Delta = D = J = S = T = 0$. coincident planes.

We might further subdivide these classes. For example, if the surface belongs to (IV) and $I = 0$, the planes are perpendicular; or if it belongs to (I), (II), (III), or (V) and $18IJD - 4I^3D + I^2J^2 - 4J^3 - 27D^2 = 0$, it is a surface of revolution. The conditions for a sphere are that it belong to (I) and $9D - IJ = 0$, $3J - I^2 = 0$.

10. *Metric properties.* Metric properties of the surface can be expressed in terms of I, J, D, Δ, S (if $D = \Delta = 0$), and T (if $S = J = D = \Delta = 0$). For example, for the surface $x^2 - 2y^2 - z^2 + 3yz + xy + 3y - 2z - 1 = 0$, we have $I = -2$, $J = -7/2$, $D = \Delta = S = 0$. This is a pair of real intersecting planes. It can be reduced to $ax^2 + by^2 = 0$, where a and b are the roots of $2x^2 + 4x - 7 = 0$. The angle between the planes can be found easily from the roots of this equation.

11. *Covariants.* Some teachers may find it preferable, particularly in more advanced courses, to define and use covariants of the curve or surface instead of the conditional invariants used here. A complete system of concomitants for (1) is composed of the invariants L, D, and Δ with two covariants, namely the left member of (1) itself and the function

$$D(x^2+y^2)+2(bg-hf)x+2(af-gh)y+ac-g^2+bc-f^2.*$$

The latter vanishes for all points from which a pair of mutually perpendicular tangents can be drawn to the curve (1). This suggests a natural method of derivation. It reduces to the conditional invariant M when $D = \Delta = 0$.

A complete system of concomitants for (11) is composed of the invariants I, J, D, and Δ, with three covariants, namely the left member of (11) itself, the function

$$G \equiv D(x^2+y^2+z^2) - 2Lx - 2My - 2Nz + S,$$

(where L, M, and N are the cofactors of l, m, and n in Δ)

$$G \equiv D(x^2+y^2+z^2) - 2Lx - 2My - 2Nz + S,$$

where L, M, and N are the cofactors of l, m, and n in Δ, and the function

*MacDuffee, loc. cit., p. 247.

$$
\begin{aligned}
H \equiv (ab+ac-h^2-g^2)x^2 &+ (ab+bc-f^2-h^2)y^2+(bc+ac-f^2-g^2)z^2 \\
&+2(af-hg)yz+2(bg-fh)zx+2(ch-fg)xy+2(bl+cl-hm-gn)x \\
&\quad +2(am+cm-fn-hl)y+2(an+bn-fm-gl)z+ID-l^2-m^2-b^2.^*
\end{aligned}
$$

G vanishes for all points from which three mutually perpendicular tangent planes can be drawn to the surface (11), and H for all points from which three mutually perpendicular tangent lines can be drawn. G reduces to $-S$ when $D=\Delta=0$, and H to $-T$ when $S=J=D=\Delta=0$.

Since lines and planes of symmetry of (1) and (11) are covariant curves and surfaces, the use of these covariants enables us to determine more easily the relation of the curve or surface to the original coordinate system.

*Paradiso, loc. cit., p. 409.

Mathematical World News

Edited by
L. J. ADAMS

The American Mathematical Society met at Columbia University on October 30. The morning session was divided into two groups, one for algebra and topology and the other for analysis and differential geometry. In the afternoon Professor D. J. Struik, Massachusetts Institute of Technology, delivered an address on *The Application of Tensor Analysis to Problems of Electrical Engineering.*

The American Mathematical Society met at the University of Iowa on November 26-27. Invited addresses scheduled were: *Singular Point Problems in the Theory of Linear Differential Equations* by Professor W. J. Trijitzinsky, University of Illinois, and *Fundamental Concepts of the Theory of Probability* by Professor A. H. Copeland, University of Michigan.

Associate Professors David Vernon Widder and Marshall Harvey Stone of Harvard University have been promoted to full professorships.

Professor H. H. Downing, University of Kentucky, contributes the following news items: Professor E. L. Rees has returned after spending last year traveling in Australia, South Africa, and some of the southern Pacific Islands. Professor C. G. Latimer goes to the University of Wisconsin for the second semester as Visiting Professor. Assistant Professor D. E. South returns after a year's study at the University of Michigan. Miss Sallie Pence returns after having received the Ph.D. in June, 1937, from the University of Illinois. Mr. C. W. Williams, graduate assistant 1936-1937, has accepted a position at Washington and Lee. Mr. J. E. Davidson, A. B. (mathematics major), Kentucky Wesleyan, has accepted a graduate assistantship at the University of Kentucky for 1937-1938.

Mr. K. L. Palmquist and Mr. O. B. Ader received their Ph. D. degrees at the University of Kentucky, August, 1937, and have accepted positions at Southern Methodist University, Dallas, Texas, Their thesis subjects were respectively, *Ideals in a Quarternion Algebra and Hermitian Forms* and *Concerning Affine Invariants of Convex Regions.*

On October 29, 1937, Dr. Archibald Henderson, head of the mathematics department in the University of North Carolina, delivered two addresses before the East Tennessee Education Association at Knoxville: at 10:00 a. m. on the subject, *Newton, Einstein, and the Theory of Relativity*; and at 2:30 p. m., before the Mathematics Section, on *Some Intimate Interrelations of Elementary and the Higher Mathematics.*

Professor Charles F. Thomas sends news items from the Case School of Applied Science. The first meeting of the Case Mathematical Club was held at the home of Dean T. M. Focke, on Friday evening, October 22. Professor O. E. Brown spoke on the subject, *The Application of Determinants and Projective Transformation to Alignment Charts.* Dr. R. F. Rinehart, formerly of Ashland College, has joined the Teaching Staff. In June, Professors Brown, Burington, and Justin attended the Cambridge Meeting of the Society for the Promotion of Engineering Education.

Professor Wilhelm Lorey, of Frankfort a. M., is carrying on research at the University of Jena on the history of mathematics since 1588. On June 17 he lectured before the Jena Mathematical Society on *Modern Mathematical Processes in Statistics and Insurance,* and in September at Kreuznach before the German Physicists' and Mathematicians' Day on *Preliminary Remarks on the Study of Mathematics in the 16th, 17th and 18th Century.*

Professor Edward Hiram McAlister of the department of mathematics of Oregon State College retired in January, 1937, after forty-two years of teaching service. In June, 1937, Professor McAlister was awarded the honorary degree of Doctor of Science by the University of Oregon, where he had taught for many years.

Dr. W. J. Kirkham of the staff of the mathematics department of Oregon State College is on a year's leave of absence. Dr. Kirkham is in Washington, D. C., engaged on statistical work for the Government.

Professor W. E. Milne, head of the department of mathematics, Oregon State College, informs us that three new men have been added to the staff of the department of mathematics at Oregon State College: Dr. Henry Scheffe from the University of Wisconsin; Dr. Paul G. Hoel from Rose Polytechnic Institute, Terre Haute, Indiana; and Dr. Orville G. Harrold of Stanford University.

Problem Department

Edited by
ROBERT C. YATES

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to Robert C. Yates, College Park, Maryland.

SOLUTIONS

No. 147. Discussion by *W. V. Parker*, Louisiana State University.

Regarding the necessity of condition (1):*

From $\quad 2x^5=a^4+b^4+c^4$, we get $4x^{10}=(a^4+b^4+c^4)^2$,

$\quad 2x^{10}=a^8+b^8+c^8$, we get $4x^{10}=(a^8+b^8+c^8)$.

Therefore,
$$(a^4-b^4-c^4)^2=4b^4c^4,$$
$$a^4-b^4-c^4=\pm 2b^2c^2,$$
$$a^4=(b^2\pm c^2)^2,$$
$$a^2=\pm(b^2\pm c^2).$$

That is, a,b,c are the sides of a right triangle. If c is to denote the largest of the three numbers, then we have $c^2=a^2+b^2$. Hence we may write:

$$a=n(\alpha^2-\beta^2),\quad b=2n\alpha\beta,\quad c=n(\alpha^2+\beta^2)$$

where α and β are relatively prime and one is even and the other odd.

Regarding the completeness of the solutions of Mahrenholz: If x_1, a_1, b_1, c_1, is a solution, then obviously k^4x_1, k^5a_1, k^5b_1, k^5c_1 is also a solution. His formulas do not give all solutions. In order to obtain the solution $x=3^4\cdot 481$, $a=3^5\cdot 1443$, $b=3^5\cdot 1924$, $c=3^5\cdot 2405$, for in-

*See page 49 of the October issue N. M. M., this volume.

stance, it would be necessary that $\alpha^2-\beta^2=9$, $2\alpha\beta=12$, $\alpha^2+\beta^2=15$ which is impossible.

Let us call a solution primitive, then, if a,b,c do not have a common factor of the form k^5 where k is an integer. Do his formulas give *all primitive solutions*? Since n is a factor of a, b, and c, if the solution is primitive every prime factor of n occurs as a factor of degree less than five. Suppose n contains the factor p^s, ($s<5$; p a prime) then $4s$ is prime to 5, and $\alpha^8+14\alpha^4\beta^4+\beta^8$ must contain the factor p^t where $4s+t\equiv 0 \pmod 5$ or $t\equiv s \pmod 5$, since

$$n^4(\alpha^8+14\alpha^4\beta^4+\beta^8)=x^5.$$

That is, $t=s+5r$ where r is zero or a positive integer. Therefore n is a factor of $\alpha^8+14\alpha^4\beta^4+\beta^8$ and hence $\alpha^8+14\alpha^4\beta^4+\beta^8=n\cdot y^5$ where y is an integer and consequently $x=ny$. The solutions given by Mahrenholz are those for which $y=1$. *Do there exist integers α,β relatively prime with one even and the other odd such that $\alpha^8 14\alpha^4\beta^4+\beta^8\equiv 0 \pmod{y^5}$, $y\neq 1$?* If such integers exist then $x=ny$, $a=n(\alpha^2-\beta^2)$, $b=2n\alpha\beta$, $c=n(\alpha^2+\beta^2)$ where $n=(\alpha^8+14\alpha^4\beta^4+\beta^8)/y^5$ is a solution which is primitive if y is chosen so that n does not contain a prime factor to a power greater than four. Such a solution would not be given by the formulas, for if λ,μ are relatively prime integers one even and the other odd such that

$$\lambda^8+14\lambda^4\mu^4+\mu^8=ny,\ (\lambda^2-\mu^2)(\lambda^8+14\lambda^4\mu^4+\mu^8)=n(\alpha^2-\beta^2)$$

$$2\lambda\mu(\lambda^8+14\lambda^4\mu^4+\mu^8)=2n\alpha\beta,\ (\lambda^2+\mu^2)(\lambda^8+14\lambda^4\mu^4+\mu^8)=n(\alpha^2+\beta^2)$$

then $\lambda^2-\mu^2=(\alpha^2-\beta^2)/y$, $\lambda\mu=\alpha\beta/y$, $\lambda^2+\mu^2=(\alpha^2+\beta^2)/y$ but $\alpha^2-\beta^2$, $\alpha\beta$, $\alpha^2+\beta^2$ do not have a common factor.

In any case we may say that all solutions are given by x,a,b,c of these formulas or by $(k/y)^4x$, $(k/y)^5a$, $(k/y)^5b$, $(k/y)^5c$.

The foregoing question, however, has not been answered. Note that

$$\alpha^8+14\alpha^4\beta^4+\beta^8=(\alpha^4-\beta^4)^2+(4\alpha^2\beta)^2=(\alpha^4+\beta^4)^2+3(2\alpha^2\beta^2)^2$$

and hence that y cannot have a prime factor of the form $(4n-1)$ nor a prime factor of the form $(6n-1)$. (See *Carmichael*, Diophantine Analysis, pp. 39 and 67.)

No. 157. Proposed by *B. D. Mayo*, Virginia Military Institute.

The distances of the three nearest vertices of a square from a given fixed point are a,b,c where $a\neq b\neq c$. Construct the square.

Solution by *E. P. Starke*, Rutgers University.

Let the fixed point be P, let the vertices to which the distances a, b, c are measured be A, B, C, respectively and let s be the length of the side of the square. To fix our ideas, let the fourth vertex D be opposite A. Since P is nearer to B than to D; then A, B, and P lie on the same side of the perpendicular bisector of AC. Similarly A, C, and P lie on the same side of the bisector of AB. Thus $a<b$ and $a<c$.

Applying the law of cosines to the triangle BPC and remembering that the diagonal is $s\sqrt{2}$, we have

$$(1) \qquad 2s^2=b^2+c^2-2bc\cdot\cos(BPC), \text{ angle } BPC>\pi/2.$$

Let the projections of AP on AB and AC be x and y, respectively. Then

$$x^2+y^2=a^2, \quad (s-x)^2+y^2=b^2, \quad x^2+(s-y)^2=c^2.$$

If we eliminate x and y from these equations and solve the results for $2s^2$, we have:

$$(2) \qquad 2s^2=b^2+c^2\pm[4b^2c^2-(b^2+c^2-2a^2)^2]^{1/2}.$$

Comparison of (1) and (2) yields easily the relation

$$(3) \qquad \sin(BPC)=(b^2+c^2-2a^2)/2bc.$$

If now we construct a triangle having sides $a\sqrt{2}$, b, and c,* the cosine of the angle opposite $a\sqrt{2}$ equals the right member of (3); thus if this angle is increased by $\pi/2$, we shall have angle BPC; finally the triangle with sides $PB=b$, $PC=c$, and included angle BPC given by (3) has for its third side BC, the diagonal of the required square, and the completion of the square is then obvious.

No. 158. Proposed by *William E. Byrne*, Virginia Military Institute.

In many texts the following problem appears:

"Prove the identity

$$2\cdot\text{arc tan } x=\text{arc sin } 2x/(1+x^2)."$$

What are precise relations between the two functions of x when principal values of the angles are used? That is, when

$$-\pi/2<\text{arc tan } x<\pi/2 \text{ and } -\pi/2\leq\text{arc sin } x\leq\pi/2.$$

Solution by *E. P. Starke*, Rutgers University.

We have: $\sin 2\theta=2\sin\theta\cos\theta=2\tan\theta/\sec^2\theta$.

*From (3) since $\sin(BPC)<1$, we have $|b-c|<a\sqrt{2}$; hence if the required square is possible, this triangle can be constructed. Also, from the original equations in x and y, we have $2a^2<b^2+c^2$ so that the right member of (3) is always found to be positive.

Therefore $$\sin(2\text{arc tan } x) = 2x/(1+x^2)$$

for all values of arc tan x. Thus $2\cdot$arc tan x always equals arc sin $2x/(1+x^2)$ if proper values are chosen for the functions of x. If angles are restricted to principal values and $|x| \leq 1$, the foregoing statement is still true. But if $|x| > 1$, the correction for the principal values gives the following:

$$\text{if } x>1, \quad 2\cdot\text{arc tan } x = \pi - \text{arc sin } 2x/(1+x^2);$$

$$\text{if } x<-1, 2\cdot\text{arc tan } x = -\pi - \text{arc sin } 2x/(1+x^2).$$

Also solved by the proposer.

No. 160. Proposed by *E. P. Starke*, Rutgers University.

Show that the sum of the exponents of the highest power of 2 not greater than $2n$, $2n/3$, $2n/5, \cdots$, respectively, equals n. Can this be generalized?

Solution by the *Proposer*.

Let $E_s(m)$ represent the exponent of the highest power of s not greater than m.* Then $n = \sum E_s(sn/i)$, where i runs through all integral values less than sn which are not divisible by s.

Classify the integers not greater than sn into $(s-1)\cdot n$ sets according to their greatest divisor not divisible by s, thus:

$$1, s, s^2, \cdots; 2, 2s, 2s^2, \cdots \text{ (if } s \neq 2); \text{etc.};$$

the last set will contain the single integer, $(sn-1)$. Now the first class contains $E_s(sn)+1$ integers; the second, $E_s(sn/2)+1$ integers etc.; the last

$$E_s[sn/(sn-1)]+1=1$$

integer. Hence since each of the sn integers considered is in some set,

$$\sum E_s(sn/i) + (s-1)\cdot n = sn,$$

which reduces at once to the stated relation.

No. 161. Proposed by *Frank Morley*, Baltimore, Maryland.

It is known that parallels to the sides of a triangle ABC through the symmedian point K meet the sides on a circle. Prove that these parallels meet the tangents at A, B, C, to the circumcircle in points on a circle.

*Thus, if $s^c \leq m < s^{c+1}$, we have $E_s(m) = c$.

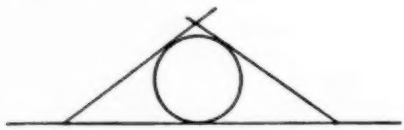


Fig. 1.

Solution by the *Proposer*.

Take the circumcircle as base circle. A concentric circle, (0), will meet the tangents at points $A(1=\lambda i)$, $A(1+\lambda i)$* and so on, where λ is positive and $i=\sqrt{(-1)}$.

*Here A, B, C, are used as complex numbers to define the triangle: $A=A_1+iA_2$, etc.—Editor.

Take a diagonal of this hexagon, say the join of $B(1-\lambda i)$ and $C(1+\lambda i)$. This has the equation

$$x+BC\cdot\bar{x}=B(1-\lambda i)+C(1+\lambda i).$$

If the three such relations are to be consistent, then by adding:

$$3x+S_2\bar{x}=2S_1,$$

and this equation defines the symmedian. For, in general, points x and y are "isogonal conjugates" when

$$x+y+S_2\cdot\bar{x}\cdot\bar{y}=S_1$$

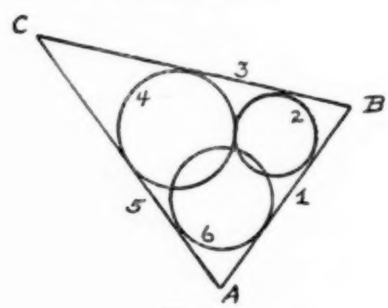


Fig. 2.

and the symmedian is the isogonal conjugate of the centroid, $y=S_1/3$. The point of the matter is that under an inversion, with center K, which sends (0) into itself, we get Fig. 2. Here is a hexagon of arc 1, 2, $\cdots$6, touching in succession. Arcs 1 and 4, 2 and 5, 3 and 6 pair off in an inversion. The vertices accordingly are on a circle, 1, 3, 5, meeting at∞, 2, 4, 6 at K.

Solution by ***Walter B. Clarke***, San Jose, California.

Let O be the circumcenter of triangle ABC; XY the tangent to the circumcirlce at A; E, F the intersections of the Lemoine circle with the sides b and c which are nearest to vertex A.

Now KE, KF cut XY at G, H, where K is the given symmedian point. By the conditions of the problem, $AFKE$ is a parallelogram with diagonals bisecting each other at J. From the proof of the theorem that parallels to the sides of ABC through K cut a,b,c in 6 cyclic points we know that EF is antiparallel to BC with respect to A.

Since the altitude from A to a is the symmetric of AO with respect to the bisector of A it follows that AO will make the same angle with EF as the altitude makes with BC, i. e. EF is perpendicular to AO and therefore EF is parallel to XY.

Since $KJ=JA$, then $KF=FH$ and $KE=EG$. The same proof will hold for other pairs of intersections on either side of B and C.

If we take K as a center of symmetry with a ratio 2:1 it follows, since E, F, etc., are cyclic, that G, H, etc., are also cyclic. As the center of the Lemoine circle is at the mid-point of KO, so the center of the GH circle is O, the circumcenter of ABC.

No. 163. Proposed by *Walter B. Clarke*, San Jose, California.

Two rigid rods of indefinite length are hinged to a straight track at points 6 feet 2 inches apart. A wheel 2 feet in diameter is placed midway between the rods and rolled along the track with the rods constantly touching it. (a) How far must the wheel roll until the rods are parallel? (b) What is the locus of the intersection of the rods?

Solution by *E. P. Starke*, Rutgers University.

Let the track be the $X-$axis with the origin at the left-hand hinge. The equation of a circle with center at $(h,1)$ and radius 1 is

$$(x-h)^2+(y-1)^2=1.$$

(a) Let the equations of the rods when parallel be $y=mx$ and $y=m(x-2a)$, where $2a=37/6$ and $m\neq 0$. If we substitute each value of y in the circle equation and set the discriminant of the resulting quadratic equal to zero (the condition for tangency), we shall have

$$m=2h/(h^2-1) \text{ and } m=2(h-2a)/[(h-2a)^2-1],$$

respectively. When these values are equated, we obtain

$$h^2-2ah+1=0 \text{ or } h=a\pm(a^2-1)^{1/2}.$$

As the circle started from center $(a,1)$, the distance rolled is $(a^2-1)^{1/2}$ which is 35/12 or 2 feet 11 inches.

(b) The equations of the rods are

$$y=2hx/(h^2-1) \text{ and } y=2(h-2a)(x-2a)/[(h-2a)^2-1].$$

The elimination from these equations of the parameter h is not difficult and yields

$$(x-a)^2=[a^2(y-2)-y]\cdot(y-1)^2/(y-2).$$

This is a cubic with double-point at $(a,1)$; intercepts on the X-axis at $x=0$, $2a$; asymptote $y=2$; and symmetric in the line $x=a$. The simple branch above the asymptote has a minimum point at $[a, 2a/(a^2-1)]$ or (37/12, 2738/1225). This upper branch is the only one that occurs unless the rods extend indefinitely in both directions through the hinges. Further, to get the lower branch of the cubic, we must demand that neither rod remain in the position of the X-axis.

Also solved by the *Proposer.*

No. 165. Proposed by *V. Thébault*, Le Mans, France.

With the digits 0,1,2,3,4,5,6,7,8,9, taken once, form five numbers such that they may be arranged in an arithmetic progression.

Solution by *E. P. Starke*, Rutgers University.

Let a, $a+d, \cdots$, $a+4d$ be the progression. Evidently $d \geq 8$, otherwise at least one pair of consecutive terms have the same initial digit; also d is not a multiple of 5, otherwise duplication occurs in the final digits. The sum of the terms, $(5a+10d)$, is divisible by 9 since the sum of the digits is. Thus

(1) $a+2d$ is a multiple of 9.

As there are ten digits to be used, the terms must consist of (a) two digits each or (b) one, two, two, two, and three digits. By consideration of the fifth term, we must have in the respective cases:

(2) $a+4d<100$, $a\geq 10$; $a+3d<100$, $a+4d>100$, $a<10$.

We may now take $d=8, 9, 10, \cdots, 33$, compute the corresponding possibilities for a and examine as much of the resulting progression as may be necessary. Thus, if $d=8$, then by (1) $a=2$, 11, 20, 29, 38, 47, 56, or 65; larger values of a contradict (2). These values all fail since some duplication of digits is found in each progression obtained. Proceeding thus with each admissible value of d, we find four solutions:

$$(a,d)=(10,22), (18,18) (50,11), (54,9).$$

In all, 61 sets (a,d) were examined. If it were permissible to use two digits to represent $a<10$, there would be four more solutions for the pair (a,d), viz.

$$(01,22),\quad (05,11),\quad (09,9),\quad (09,18).$$

Also solved by the *Proposer* and partially so by *Albert Farnell.*

No. 166. Proposed by *Karleton W. Crain,* Purdue University.

Find and describe the envelope of the Euler line of the triangle ABC, where B and C are fixed and A moves along the line AC.

Solution by the *Proposer.*

Let the rectangular coordinates of the vertices be $A(a,0)$, $B(0,b)$, $C(c,0)$. Then the equation of the Euler line is:

$$F(x,y,a)\equiv(b^2+3ac)x-b(a+c)y-ac(a+c)=0. \tag{1}$$

Differentiating,

$$dF/da=3cx-by-c(2a+c)=0.$$

Eliminating a between these equations, the equation of the required envelope is:

$$9c^2x^2-6bcxy+b^2y^2+(4b^2c-6c^3)x-2bc^2y+c^4=0. \tag{2}$$

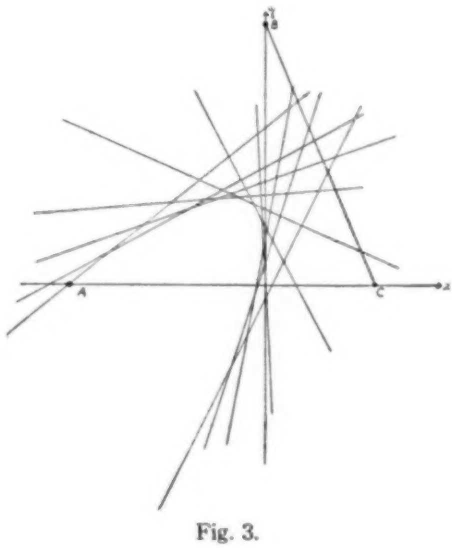


Fig. 3.

Since the second degree terms form a perfect square and the discriminant of the equation is not zero, the envelope is a parabola and not two parallel lines.

The parabola is tangent to the Y-axis at the point where $y=c^2/b$, for when $a=-c$, equation (1) becomes $x=0$, and if $x=0$ in equation (2), $y=c^2/b$.

In the special case where $b=c\sqrt{3}$, (i. e., where angle $BCA=60°$) the parabola is tangent to the Y-axis where $y=c\sqrt{3}$ and to the X-axis where $x=-c/3$.

No. 168. Proposed by *H. D. Grossman*, New York City.

N coins are tossed and all that fall heads are removed. The remaining ones are tossed again after which all heads are again removed. How many times must this operation be repeated if the probability that no coins remain is to be at least $\frac{1}{2}$?

Solution by *E. P. Starke*, Rutgers University.

After one trial with M coins, the probability of j coins falling "tails", and thus remaining, is

$$\binom{M}{j}\Big/2^m, \text{ where } \binom{M}{j}$$

is the familiar binomial coefficient and

$$\binom{M}{0}=1.$$

It can be shown that the probability that exactly j coins of the original N remain after p operations have fallen "tails" in every trial is

$$\binom{N}{j}(2^p-1)^{N-j}\big/2^{pN}. \tag{1}$$

This is most easily established by induction on p. Assume then that the probability of k coins still remaining after $p-1$ operations is

$$\binom{N}{k}(2^{p-1}-1)^{N-j}\big/2^{(p-1)N}.$$

The probability that j of these k coins will remain through the pth operation is

$$\binom{k}{j}\Big/2^k.$$

Hence the probability of exactly j coins after p operations is

$$\sum_{k=j}^{N}\binom{k}{j}\binom{N}{k}(2^{p-1}-1)^{N-k}/2^{(p-1)N+k},$$

which can be reduced to

$$\sum_{k=j}^{N}\binom{N}{j}\binom{N-j}{k-j}(2^{p-1}-1)^{N-k}/2^{(p-1)N+k}$$

$$=\binom{N}{j}2^{-pN}\sum_{k=j}^{N}\binom{N-j}{k-j}(2^{p}-2^{1})^{N-k}$$

$$=\binom{N}{j}2^{-pN}(2^{p}-2+1)^{N-j}$$

which is (1). Since (1) is true also for $p=1$, the induction is complete.

Now in particular if $j=0$, we are to determine p such that the probability shall be at least $\frac{1}{2}$. Hence

$$(2^p-1)^N\cdot 2^{-pN}\geqq\tfrac{1}{2}.$$

Thus p is the smallest integer not less than

$$\log_2[2^{1/N}/(2^{1/N}-1)].$$

Also solved by the *Proposer*.

PROPOSALS

Note: The reader is urged to submit solutions or discussion for the following unsolved problems:

Nos. 30, 48, 58, 62, 107, 125, 128, 129, 132, 133, 134, 146, 150, and 155.

No. 183. Proposed by *G. W. Wishard*, Norwood, Ohio.

Prove the following rule for finding whether any given number is divisible by 3: Pass your pencil over the given number, jumping over every 0, 3, 6, or 9, and letting it touch each 1, 4, or 7 in one place, and each 2, 5, 8 in two places. Count 1, 2, 3, 1, 2, 3, as the pencil touches. If the last count is 3, the number is divisible by 3. Otherwise, the last count will be the remainder.

No. 184. Proposed by *Albert Farnell*, Centenary College.

Integrate:

$$\int \sec^4 x \cdot \tan x \cdot \log^4 \sec x \cdot dx.$$

No. 185. Proposed by *V. Thébault*, Le Mans, France.

With the digits 0,1,2,3,4,5,6,7,8,9, taken once, form a perfect square which is divisible by 66. The solution is unique.

No. 186. Proposed by *V. Thébault*, Le Mans, France.

Consider the square $ABCD$ and the tangents to the circumcircle at two opposite vertices. $M_1, \cdots, M_6$ are the orthogonal projections of an arbitrary point, M, of this circle upon the sides of the square and the two tangents. Show that $M_1, \cdots, M_6$ are the vertices of a hexagon of area equal to that of the square and whose consecutive sides are perpendicular.

No. 187. Proposed by *V. Thébault*, Le Mans, France.

Let A_1, B_1, C_1 be the middle-points of the sides BC, CA, AB of a triangle ABC inscribed to a circle with center 0. Prolong the lines OA_1, OB_1, OC_1 to lengths $AA' = r_a/2$, $BB' = r_b/2$, $CC' = r_c/2$ where r_a, r_b, r_c are the radii of the circles escribed to ABC.

(1) Show that the circumcircles of $A'BC$, $B'CA$, $C'AB$ are tangent to the inscribed circle of ABC and to the circumcircle of $A'B'C'$.

(2) Examine the case where the radius of one of the escribed circles is replaced by that of the inscribed circle of the fundamental triangle.

No. 188. Proposed by *V. Thébault*, Le Mans, France.

In what number system can a number of four digits of the form $aabc$ be the square of a number of two digits mn, being given that $c = b+1$ and $m = n+1$?

No. 189. Proposed by *J. Rosenbaum* Bloomfield, Connecticut.

Prove that if $x^3 + y^3 = z^3$ has a solution in integers, all different from zero, then there exist integral solutions, all different from each other and from zero, for the following:

(a) Sum of three cubes equals the sum of two cubes.

(b) Sum of four cubes equals the sum of three cubes.

(c) Sum of five cubes equals the sum of four cubes.

No. 190. Proposed by *Nathan Altshiller-Court*, University of Oklahoma.

Given the tetrahedron $ABCD$ and a point L in space. The parallel to the lines LB, LC, LD, through the mid-points of the edges AB, AC, AD, respectively, have a point P in common. We have three analogous points Q, R, S, for the vertices B, C, D. Show that the lines joining the points P, Q, R, S to the centroids of the respective faces of $ABCD$ are concurrent.

No. 191. Proposed by *Nathan Altshiller-Court*, University of Oklahoma.

The tangent plane to a given sphere, center 0, at a variable point M cuts a fixed plane along a line d. Find the locus of the projection, L, of the point M upon the plane passing through 0 and d.

No. 192. Proposed by *William E. Byrne*, Virginia Military Institute.

If $u=f(x,y,z)$ may be developed about (x_0,y_0,z_0) in a Taylor series, and if

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$$

are not all zero at (x_0,y_0,z_0) the limit

$$\text{Limit } (\Delta u/du)$$

as $\Delta x \to 0$, $\Delta y \to 0$, $\Delta z \to 0$ in such a way that $du \to 0$ by values different from zero is not always 1. Find the possible limiting values and characterize them geometrically.

No. 193. Proposed by *Walter B. Clarke*, San Jose, California.

Two unlimited straight lines intersect at A. Points B, C, D, and E are taken anywhere on these two lines but B and D on one, C and E on the other. The line joining the mid-points of BC and DE intersects AB at G and AC at F. The line joining the mid-points of BE and DC intersects AB at H and AC at J. On AC take $AK=AF$ and $AL=AJ$. Show that HJ is parallel to GK and that HL is parallel to GF.

No. 194. Proposed by *Walter B. Clarke*, San Jose, California.

Given an isosceles right triangle with $a=b$. Take D on AB so that $AD=a$. Show that if another triangle is constructed with circumradius equal to AD and inradius equal to BD, circumcenter of the of the second triangle will lie on its incircle.

No. 195. Proposed by *Walter B. Clarke*, San Jose, California.

Let A be any vertex of a triangle ABC; D and H the contact points of the incircle with sides BC and AB, respectively. E is the contact point of the excircle relative to A with BC. AE cuts the incircle at F and G, in that order. Show that: (1) triangles DEG and DFG are similar; (2) $AF:FE::AH:BC$.

Reviews and Abstracts

Edited by
P. K. SMITH

Functions of Real Variables. By William Fogg Osgood. Peking, University Press, The National University of Peking, 1936. xii+399 pages.

This book is planned as a first approach to the Theory of Functions to follow a course in Advanced Calculus. The main emphasis is on new methods which are illustrated by many applications and exercises. More details are included in the elementary theorems than in the author's German treatise while many topics are omitted or referred to briefly. Recent references have been added. Incomplete proofs in Osgood's Advanced Calculus are finished. It is these features and the care with which it is written that recommend the book.

In dealing with series the term "sum" is replaced by "value". General tests for convergence and divergence are given and also a splendid appraisal of the error when the series is broken off at the mth term. A preliminary discussion of infinite products leads to the study of the convergence of hypergeometric series.

To expand the number system beyond integers, fractions are defined as number pairs (m, n) and irrational numbers are treated by Dedekind's method. It is refreshing reading although difficult for a first course.

Concise definitions and proofs are clearly presented in the introduction to limits and continuity. One improvement could be made if all parts of the hypothesis were given at the beginning of a statement.

The treatment of derivatives and integrals is an extension of that in calculus. Rolle's theorem and the Law of the Mean are proved arithmetically. The theory of functions of several variables is presented as well as a good discussion of implicit functions. Here the subscript notation for partial differentiation could be used consistently if superscripts were introduced to indicate the different functions.

Extremely suitable examples preface the work on uniform convergence and integration and differentiation of series term by term. An application of series is derived by defining $\sin x$ and $\cos x$ as power series solutions of the differential equation

$$\frac{d^2y}{dx^2}+y=0$$

The integral definition is used for log x. Finally we have an elementary deduction of

$$\pi \cot \pi x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2x}{n^2 - x^2}$$

by de Moivre's Theorem.

In continuing the study of infinite series the commutative and associative laws are considered. Then double series and series of series are treated and good examples given. The work of Fourier's series is well organized and appropriate references are inserted.

The chapter on definite, improper, and line integrals again emphasizes the caution needed in dealing with double limits. The Gamma and Beta functions are examined in detail. Fourier's integral is developed heuristically at first and an account of Fourier's integral for functions of several variables concludes the discussion.

A number of existence theorems for ordinary and partial differential equations are proved. The need for an hypothesis to forestall multiple solutions is illustrated by another well planned example.

There is no index, but the table of contents gives the paragraph headings. Typographical errors are not numerous and can be detected easily for the most part.

Institute for Advanced Study. MARIE M. JOHNSON.

The Study of the History of Mathematics. By George Sarton. Harvard University Press, Cambridge, Massachusetts. Oxford University Press, London, Humphrey Milford. 1936. 113 pages, $1.50.

The author of this work has been giving lectures in Harvard in the history of science, but, more recently, in the history of mathematics also. The present volume is the kernel of his inaugural lecture in the latter course. In it, among other things, Dr. Sarton asserts that the main duty of the historian of mathematics, as well as his fondest privilege, is to explain the humanity of mathematics, to illustrate its greatness, beauty, and dignity.

The approach to his main subject matter is through the medium of a delightful twenty-four page discourse upon the study of the history of mathematics. The reader will thoroughly enjoy the many fine sentences found therein.

Sub-topics of this first division of the book are: *Various Kinds of Mathematical History, Secret History, The Greek Miracle, Mathematical Progress, Future Mathematics, External and Internal Influences,*

Necessity and Caprice, Ways of Discovery, Historical vs. Mathematical Synthesis, Mathematical Genius, History as a Recreation, and *Dangerous Pedantry.*

Note on the Study of the History of Modern Mathematics is the title of the second division, pages 29-38. The following sentence (page 36) is characteristic: "The historian who is worth his salt will take infinite pains to collect all available evidence, sift the facts, test them, and avoid every conceivable source of error." To which Dr. Sarton adds this significant and humorous comment: "This will increase his labor considerably, and not necessarily increase his fame, for the average reader will not see the difference anyhow."

A splendid bibliography covers pages 41-65. Sub-topics are: *General Treatises, Hand-books, Treatises Devoted to ths History of Special Branches of Mathematics, Mathematics in the Nineteenth and Twentieth Centuries, Philosophy and Methodology, Bibliography, Journals on the History of Mathematics,* and *Centres of Research.*

An appendix of thirty-seven pages gives brief biographies of one hundred and eighteen modern mathematicians. Names of living persons are omitted. The arrangement is alphabetical. However, pages 99-100 re-list the same names, this time chronologically according to their death years. In each biography three things are told: (1) the best availabe biographies; (2) editions of their collected works; (3) editions of their correspondence.

Two indices close the volume—the first of "persons", the second of "institutions and serials." A good book—with a nice index to it—what a joy to a real student! This is that kind of a book with that kind of an index.

With respect to the mechanics of the book—color and grade of paper, arrangement of matter on the page, kind of type, quality of binding—of these no adverse criticism is offered. The author and the publishers have combined to produce a satisfactory piece of work.

Therefore, this volume commends itself most highly to all mathematicians. It is a convenient, time-saving, helpful little book to have right on one's desk; and becoming acquainted with it, the reader will agree with its author wherein he says, "*The study of the history of mathematics will not make better mathematicians but gentler ones. It will enrich their minds, mellow their hearts, and bring out their finer qualities.*"

Louisiana State University. IRBY C. NICHOLS.

Special Topics in Theoretical Arithmetic. By Joseph Bowden. Joseph Bowden, Garden City, New York, 1936. xi+217 pages.

According to the preface this book is a sequel to the author's *Elements of the Theory of Integers.* Since the reviewer has not had an

opportunity to examine the latter, it is impossible to judge the two books as a unit. The apparent purpose of this volume is to develop some of the more elementary properties of integers which are omitted from most college textbooks. In the derivation of these properties a very successful attempt has been made to achieve rigor of demonstration, although in this respect the book is dependent upon its predecessor for certain definitions and elementary properties, which are here assumed.

An idea of the scope of the book can be obtained from the chapter titles: *Series and Mathematical Induction*; *Scales of Notation*; *Congruence*; *Indeterminate Equations of the First Degree in Two Unknowns*; *Mathematical Recreations*. The chapter on *Scales of Notation* is particularly worthy of comment. The basis upon which "radix notation" rests is developed with much greater generality than is usual, copious illustrations of the use of scales with radix other than ten are given, and the chapter ends with a discussion of the relative advantages of the radices 2, 4, 8, 10, 12, and 16.

Certain passages in the book, particularly in footnotes, may be interpreted as insinuations that many perfectly correct texts in mathematics are incorrect. These passages make the book a somewhat dangerous work to be placed in the hands of an immature student. The author might better have either omitted these references altogether, or enlarged upon them in such a way as to point out that the difference between his conclusions and those of the texts to which he refers are due to differences in the definitions of the terms used and does not indicate incorrectness on the part of either. We call attention, for example, to two such passages.

On p. 20, footnote (1), we find

"A word of warning may here be worth while. According to the definitions of quotient of integers given above there is no quotient of 5 and 2 nor of 3 and 0. But according to the definition of quotient of rational numbers, the next kind of numbers after integers, there may be (in fact are) quotients of the rational numbers named 5 and 2 and of those named 3 and 0, these quotients being the fractions five halves and infinity.

"In trigonometry the tangent of 90° is the number infinity, although many authors say there is no such thing as the tangent of 90°."

The author fails to give his definitions of the terms *rational number*, *division* for rational numbers, *tangent*, and *infinity*, which are obviously different from those given by the "many authors" to whom he refers, and to state that it is due to this difference of definition that he finds that division by zero is possible and that the tangent of 90° exists.

On page 83, footnote (2), we find

"According to some authors 1 is not prime. Neither is it composit! In olden days they used to say that one is not a number (unus non est numerus)."

The exclamation point might better be replaced by a period. The author has exercised his undenied right to define prime in such a manner that one is a prime and zero composite, whereas the commonly given definition does not admit these two numbers as either prime or composite, but places them in classes by themselves. The author fails to point out this reason for the difference, but takes care to emphasize it by the exclamation point.

Although, from the point of view of logic, one has the right to define a thing in any manner which one finds useful, certain mathematical terms have been defined in a certain way for so long a time that they have become part of the language of the science, and a change in their meaning should be attempted only where the advantage deriving therefrom is sufficient to offsett the confusion which will, at least for a time, result from such a change. It seems to the reviewer that the author has given definitions of common mathematical terms which are non-equivalent to those commonly accepted without deriving sufficient advantages from the changes to justify them.

Throughout the book the author uses a system of simplified (phonetic) spelling which should be a joy to those who wish to reform our admittedly illogical English orthography, but which is somewhat annoying to some of us who are so steeped in the traditional manner that "through" and "one" seem more beautiful, if less logical, than "thru" and "wun". Most readers will find the author's change in mathematical symbolism more annoying, for instance his symbols for such common relations and operations as "is not equal to", "is a divisor of", "the greatest common factor of", "the least common multiple of".

The general appearance of the book is excellent, the type is clear, and the proof reading seems to have been little short of perfect, which is remarkable in view of the extremely complicated symbolism.

College of St. Thomas. L. E. BUSH.

Mathematics for the Million. By Lancelot Hogben. W. W. Norton & Co., Inc., New York, 1937. 647 pages. $3.75.

This book demands attention because in a short time it has become a best seller in England and America. As in most popularizing books,

the attempt is made to show that it is "easy" to master mathematics. We are told that mathematics is the language or grammar of size, and also the "mirror of civilization". If a grammar, one wonders whether it represents the phonology, morphology, or syntax. As a "mirror of civilization" it depends directly on social and physical environment. The reader will naturally raise the question, "Has every discovery resulted from the need of social progress?" The entire treatment of the book is from the social, statistical and political point of view, which probably explains in part why it has appealed to such large masses of readers.

Copious historical material is woven in rather loosely, along with many statements in conversational familiar tone of personal opinion, etc., which aid in maintaining the reader's interest. The closely scrutinizing reader may question some of these interpretations and alleged historical facts, but one must keep in mind that the volume is written "for the million". The danger in this type of book is that the author is too easily led to broad generalizations without qualifying such statements.

Various chapters treat arithmetic, algebra, geometry, trigonometry, spherical triangles, graphs, logarithms, calculus and statistics. There are many figures, graphs and diagrams by J. F. Horrabin; at the end of each chapter there are questions and problems for the reader to solve. It would be interesting to learn what results might occur from the use of this book in a course in general mathematics. Each subject is developed from its historical origins and this method may prove fruitful in the future, provided it does not require too much time on the part of pupil and teacher. There is a strong tendency today to stress in elementary work the practical (and social) aspects or values of mathematics even more than formerly.

This book is excellent in format, printing, paper, etc. There are six pages of tables at the close, but no index.

University of Illinois. G. WALDO DUNNINGTON.

The Elements of Analytic Geometry. By H. E. Buchanan and G. E. Wahlin. Farrar and Rinehart. New York. 1937. 256 pages.

This book exemplifies the recent trends in analytics texts and has some innovations of its own. Polar coordinates are introduced early. Special attention is given to directed line segments. The material on tangents and normals is preceded by an introductory account of the derivative. Empirical equations and plotting with logarithmic and

semilogarithmic paper are treated at some length. The fundamental concepts and methods of solid analytics are presented in detail, almost one-third of the entire book being reserved for this part.

The book has certain distinctive features. In many of the sets of exercises there are starred problems for the superior students. Some of these starred exercises are designed to stimulate collateral reading. It seems to the reviewer that the latter is an admirable idea. The historical background of a branch of mathematics is essential to complete the presentation of the mathematics involved. The inclusion of exercises requiring supplementary reading is certainly a step to be commended.

The introductory chapter includes, among other things, the definitions of trigonometric ratios, the quadratic equation and determinants.

The problems in the chapter on Empirical Equations have been lifted bodily from practical situations in engineering and biology, and will appeal to the practical-minded student.

The order of topics of the core of the book is traditional: graphs, and loci, straight line, circle, central conics, tangents and normals, general equations and loci.

The chapter on central conics opens with a discussion of translation of axes, after which the general equation of the conic,

$$(1-e^2)x^2+y^2-2px+p^2=0,$$

is considered. Then the central conics are studied from this general equation as a point of departure.

The treatment of solid analytics is characterized by an evident effort to make the work depend upon the algebraic equations and formulas, rather than upon the geometrical figures.

This textbook is worth examining, because it is well written, and it offers a rather unique treatment and mode of presentation. The publishers have done their part well.

Santa Monica Junior College. L. J. ADAMS.

Mathematical literature received by this department:

a) *An Introduction to Projective Geometry,* by C. W. O'Hara and R. Ward, Oxford University Press, N. Y.

b) *The Duomal System of Numeration and Computation,* by Wm. E. Block, Lake View High School, Chicago, Ill.

c) *Calculate by Eights, Not by Tens,* by E. M. Tingley, 221 No. Cuyler Ave., Oak Park, Ill.

d) *Scripta Mathematica Forum Lectures,* by Cassius Jackson Keyser, D. E. Smith, E. Kasner, and W. Rautenstrauch.

e) *Matematica Elemental,* (Revista Dedicada a los Estudiantes de Matematicas, Publicada Bajo los Auspicios de la Sociedad Matematica Espanola y del Ministerio de Instruccion Publica) Tomo VI, Numero 1-6, Enero Junio 1937 Madrid, Spain.

f) *Revista Matematica Hispano-Americana,* 2 Serie, Tomo XII, Enero Junio 1937, Nums. 1-6. Madrid.

g) *Arithmetic of the Alternating,* Multifoliate Monograph E2, by Robert Ashby Philip, Monograph Press, Fairhaven, Mass.

h) *Carl Friedrich Gauss,* (Inaugural Lecture on Astronomy and Papers on the Foundations of Mathematics), Translated and edited by G. Waldo Dunnington, Louisiana State University Press, \$1.00.

i) *Solution of the Elastic Equation,* $M = EI \dfrac{dy^2}{dx^2}$, by H. B. Compton and C. O. Dohrenwend, Rensselaer Polytechnic Institute, Troy, N. Y.

j) *Groups of Finite Order,* by R. D. Carmichael, Ginn and Company.